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臨界指数の重みに関する極限吸収原理と SOMMERFELD の放射条件

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1. INTRODUCTION

This article is based on joint work with Michael Ruzhansky (Imperial College London). Let us first recall some basics on Helmholtz equation and its radiation conditions first, which is relevant to our main results described later in Section.

Let

$$L=-\Delta \ : \text{Laplacian on } \mathbf{R}^n,$$

and consider the **Helmholtz equation**

$$(L-\lambda)v=g, \quad (\lambda>0,\, g\in C_0^\infty).$$

Note that the solution is not unique because the eigenfunction of Dirichlet problem exists. At least the inverse

$$v=(L-\lambda)^{-1}g$$

should be one of the solution, but it does not exists for $\lambda>0$ since

$$R(z)=(L-z)^{-1} \ : \text{resolvent}$$

is defined only for $z\in\mathbf{C}\setminus[0,\infty)$ as the element of $\mathcal{L}(L^2,L^2)$. But the following **limiting absorption principle** holds:

Theorem 1 (Agmon [1]). *The "weak limit"*

$$R(\lambda\pm i0)=\lim_{\varepsilon\searrow 0}R(\lambda\pm i\varepsilon)$$

exists in $\mathcal{L}(L^2_k,L^2_{-k})$ for $k>1/2$. Here L^2_k is the set of functions g such that the norm

$$\|g\|_{L^2_k}=\left(\int\left|\langle x\rangle^k g(x)\right|^2dx\right)^{1/2}; \quad \langle x\rangle=\left(1+|x|^2\right)^{1/2}$$

is finite.

Then $v=R(\lambda\pm i0)g$ is a unique solution to Helmholtz equation under Sommerfeld's **radiational condition**

$$\begin{cases} v=O(r^{-(n-1)/2}), \\ (\partial_r\mp i\sqrt{\lambda})v=O(r^{-(n+1)/2}) \end{cases}$$

for $r=|x|\to\infty$, or more weakly

$$\begin{cases} v\in L^2_{-k} \quad (k>1/2) \\ (\partial_r\mp i\sqrt{\lambda})v\in L^2_{-1/2+l} \quad (l<1). \end{cases}$$

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Then we call

$$u = R(\lambda + i0)f \ : \textbf{outgoing} \text{ solution,}$$
$$u = R(\lambda - i0)f \ : \textbf{incoming} \text{ solution.}$$

In other words, the radiation condition distinguishes *outgoing* and *incoming* solutions.

The limiting absorption principle is justified by the resolvent estimate

$$\sup_{\varepsilon>0} \|R(\lambda \pm i\varepsilon)g\|_{L^2_{-k}} \leq C_\lambda \|g\|_{L^2_k}$$

for $\lambda > 0$ and $k > 1/2$. More generally, we have

$$\|D^\alpha R(\lambda \pm i0)g\|_{L^2_{-k}} \leq C_\lambda \|g\|_{L^2_k}$$

for $\lambda \neq 0$, $|\alpha| \leq 2$, and $k > 1/2$.

Furthermore, if $n \geq 2$, we have the uniform resolvent estimates

$$\sup_{\lambda \in \mathbf{R}} \| |D| R(\lambda \pm i0)g\|_{L^2_{-k}} \leq C \|g\|_{L^2_k}, \tag{1.1}$$

or equivalently

$$\sup_{\lambda \in \mathbf{R}} \|\sigma(X, D) R(\lambda \pm i0) \sigma(X, D)^* g\|_{L^2} \leq C \|g\|_{L^2} \tag{1.2}$$

for $\sigma(X, D) = \langle x \rangle^{-k} |D|^{1/2}$, where $k > 1/2$. It is not true for $k \leq 1/2$.

Uniform resolvent estimates (1.1) and (1.2) are also used to show time-space estimates for Schrödinger equations. By taking partial Fourier transform F_t in t for the Schrödinger equation:

$$(i\partial_t + L)\, u(t, x) = f(t, x)$$

we have the Helmholtz equation:

$$(L - \tau)\,(F_t u)(\tau, x) = (F_t f)(\tau, x).$$

Hence we can expect that resolovent estimates imply estimates for Schrödinger equation. In fact, we have

Theorem 2 (Kato [8]). *: Uniform resolvent estimate*

$$\sup_{\lambda \in \mathbf{R}} \|A R(\lambda \pm i0) A^* g\|_X \leq C \|g\|_X$$

implies the smoothing estimate

$$\|Au(t, x)\|_{L^2(\mathbf{R}_t : X)} \leq C \|\varphi(x)\|_X$$

for the solution to the Schorödinger equation

$$\begin{cases} (i\partial_t + L)\, u(t, x) = 0, \\ \qquad\qquad u(0, x) = \varphi(x). \end{cases}$$

(Here X is a separable Hilbert space.)

Similarly, the uniform estimate

$$\sup_{\lambda \in \mathbf{R}} \|A R(\lambda \pm i0) g\|_X \leq C \|g\|_Y$$

implies the estimate

$$\|Au(t, x)\|_{L^2(\mathbf{R}_t : X_x)} \leq C \|f(t, x)\|_{L^2(\mathbf{R}_t : Y_x)}$$

for the solution to the Cauchy problem

$$\begin{cases} (i\partial_t + L)\, u(t,x) = f(t,x), \\ \qquad u(0,x) = 0. \end{cases}$$

In fact, the solution is given by the formula

$$u(t,x) = iF_\tau^{-1}R(\tau - i0)F_t f^+(t,x) + iF_\tau^{-1}R(\tau + i0)F_t f^-(t,x),$$

where $f^\pm$ is the function f multiplied by the characteristic function of the set $\{t \in \mathbf{R}\,;\, \pm t \geq 0\}$.

The objective of this article is to explain the following:

- In uniform resolvent estimates (1.1) and (1.2), the critical case $k = 1/2$ or more general combination of the order for the weight can be attained if we assume a structure on $\sigma(X, D)$.
- The structure is related to a radiation condition.
- Such consideration is useful in the nonlinear problems in Schrödinger equations.

2. Main results

We generalise the operator to understand the meaning of the structure well, which will clarify and exhibit geometric quantities responsible for resolvent estimates in the critical case. We set

$$L = a(D) = F^{-1}a(\xi)F$$

The case $a(\xi) = |\xi|^2$ corresponds to the usual Laplacian. We always assume:

- $a(\xi) \in C^\infty(\mathbf{R}^n \setminus 0), \quad a(\xi) > 0.$
- $a(\lambda\xi) = \lambda^2 a(\xi) \quad (\lambda > 0\,, \xi \neq 0).$
- the Gaussian curvature of

$$\Sigma_a = \{\xi \in \mathbf{R}^n : a(\xi) = 1\}$$

  never vanishes.

Let $\{(x(t), \xi(t)) : t \in \mathbf{R}\}$ be the classical orbit associated to $L = a(D)$, that is, the solution of the ordinary differential equation

$$\begin{cases} \dot{x}(t) = (\nabla a)(\xi(t)), \quad \dot{\xi}(t) = 0, \\ x(0) = 0, \quad \xi(0) = k, \end{cases}$$

and consider the set of the path of all classical orbits

$$\begin{aligned} \Gamma_a &= \{(x(t), \xi(t)) : t \in \mathbf{R},\ k \in \mathbf{R}^n \setminus 0\} \\ &= \{(\lambda\nabla a(\xi), \xi) : \lambda \in \mathbf{R},\ \xi \in \mathbf{R}^n \setminus 0\} \\ &= \{(x,\xi) \in T^*\mathbf{R}^n \setminus 0 : x \wedge \nabla a(\xi) = 0\}. \end{aligned}$$

Here the outer product $a \wedge b$ of vectors

$$a = (a_1, a_2, \ldots, a_n), \; b = (b_1, b_2, \ldots, b_n)$$

is defined by

$$a \wedge b = (a_i b_j - a_j b_i)_{i<j}.$$

For example, in the Laplacian case $a(\xi) = |\xi|^2$, we have

$$\Gamma_a = \{(x,\xi) \in T^*\mathbf{R}^n \setminus 0 : x \wedge \xi = 0\}.$$

Furthermore we use the notation

$$\sigma(x,\xi) \sim \langle x\rangle^a |\xi|^b \iff \begin{cases} \sigma(x,\xi) \in C^\infty(\mathbf{R}_x^n \times (\mathbf{R}_\xi^n \setminus 0)), \\ \sigma(x,\lambda\xi) = \lambda^b \sigma(x,\xi)\,;\ (\lambda > 0, \xi \neq 0), \\ |\partial_x^\alpha \sigma(x,\xi)| \le C_\alpha \langle x\rangle^{a-|\alpha|}|\xi|^b. \end{cases}$$

We simply write $\sigma(x,\xi) \sim |\xi|$ if $a = 0$ and $b = 1$. Then we have the following main theorems:

Theorem 3. *Suppose $\sigma(x,\xi) \sim \langle x\rangle^{-1/2}|\xi|^{1/2}$ and*

$$\sigma(x,\xi) = 0 \ \textit{on}\ \Gamma_a.$$

Then we have

$$\sup_{\lambda\in\mathbf{R}} \|\sigma(X,D)R(\lambda \pm i0)\sigma(X,D)^* g\|_{L_l^2} \le C\|g\|_{L_l^2}$$

for $-1 < l < 1$ if $n \ge 3$ and $-1/2 < l < 1/2$ if $n = 2$.

Remark 1. *For uniform resolvent estimate* (1.2)*, this theorem says that structure allows the critical case $k = 1/2$ even on more general spaces $X = L_l^2$.*

Theorem 4. *Suppose $\sigma(x,\xi) \sim |\xi|$ and*

$$\sigma(x,\xi) = 0 \ \textit{on}\ \Gamma_a.$$

Then we have

$$\sup_{\lambda\in\mathbf{R}} \|\sigma(X,D)R(\lambda \pm i0)g\|_{L^2_{-1/2+l}} \le C\|g\|_{L^2_{1/2+l}}$$

for $0 < l < 1$ if $n \ge 3$ and $0 < l < 1/2$ if $n = 2$.

Remark 2. *Suppose just $\sigma(x,\xi) = |\xi|$ without structure. Then we have only*

$$\sup_{\lambda\in\mathbf{R}} \|\sigma(X,D)R(\lambda \pm i0)g\|_{L^2_{-1/2-l}} \le C\|g\|_{L^2_{1/2+l}} \quad (l > 0)$$

by estimate (1.1).

3. Herbst-Skibsted's resolvent estimate

We will explain the relation between the structure condition in our main results and Sommerfeld's radiation condition.

Let $S(x,\lambda)$ be the solution of eikonal equation,

$$a(\nabla S(x,\lambda)) + V(x) = \lambda \quad (\lambda > 0)$$

for $L = a(D) + V(x)$. (We take $V = 0$ in our context.) Let $p = -i\nabla$ be the momentum operator and set

$$\gamma(\lambda) = p \mp \nabla S(x,\lambda).$$

The quantization of $\gamma(\lambda)$ is given by

$$\bar{\gamma} = p \mp \nabla S(x, a(D)).$$

Theorem 5 (Herbst-Skibsted [6]). *Let $L = -\Delta(+V)$. Then, for $l < 1$, we have the (quantum) result*

$$\sup_{\lambda \in \mathbf{R}} \|\bar{\gamma}\, R(\lambda \pm i0)\chi(|D|)g\|_{L^2_{-1/2+l}} \leq C\|g\|_{L^2_{5/2-l}}$$

where $\chi \in C_0^\infty(\mathbf{R}_+)$. The (classical) result

$$\gamma(\lambda)R(\lambda \pm i0) \in \mathcal{L}(L^2_{5/2-l}, L^2_{-1/2+l})$$

by Isozaki [7] can be derived from this quantum result.

Note that $\|g\|_{L^2_{1/2+l}} \leq \|g\|_{L^2_{5/2-l}}$ and the estimate in Theorem 4 is a better one in this sense.

In the case $L = -\Delta$, we have

$$\gamma(\lambda) = -i\nabla \mp \sqrt{\lambda}\frac{x}{|x|}, \quad \bar{\gamma} = D \mp \frac{x}{|x|}|D|.$$

Then we remark

$$i\frac{x}{|x|} \cdot \gamma(\lambda) = \partial_r \mp i\sqrt{\lambda}.$$

Hence the classical result in Theorem 5 implies

$$(\partial_r \mp i\sqrt{\lambda})R(\lambda \pm i0) \in \mathcal{L}(L^2_{5/2-l}, L^2_{-1/2+l})$$

(Sommerfeld's radiation condition).

We also remark that the symbol $\sigma(x, \xi)$ of the operator $\bar{\gamma}$ satisfies the half structure

$$\sigma(x, \xi) = 0 \text{ on } \Gamma_a^\pm,$$

where

$$\Gamma_a^\pm = \{(\lambda\nabla a(\xi), \xi) : x \in \mathbf{R}^n \setminus 0, \pm\lambda > 0\}.$$

The quantum result in Theorem 5 means that each half structure implies the estimates for $R(\lambda + i0)$ and $R(\lambda - i0)$ respectively, while our result Theorem 4 means that full structure implies both.

4. Estimates for Schrödinger equations

As we have already discussed in Section 1, resolvent estimates automatically imply estimates for the (generalised) Schrödinger equation. By Theorem 3, we have

Corollary 6. *Suppose $\sigma(x, \xi) \sim \langle x\rangle^{-1/2}|\xi|^{1/2}$ and*

$$\sigma(x, \xi) = 0 \text{ on } \Gamma_a.$$

Then the solution to the homogeneous equation

$$\begin{cases} (i\partial_t + L)\, u(t, x) = 0, \\ \qquad u(0, x) = \varphi(x) \end{cases}$$

has the estimate

$$\left\|\langle x\rangle^l \sigma(X, D)u\right\|_{L^2(\mathbf{R}_t \times \mathbf{R}^n_x)} \leq C\left\|\langle x\rangle^l \varphi\right\|_{L^2(\mathbf{R}^n)}$$

for $-1 < l < 1$ if $n \geq 3$ and $-1/2 < l < 1/2$ if $n = 2$.

Remark 3. *This estimate with $l = 0$ is a critical case of the smoothing estimate*

$$\left\|\langle x\rangle^{-s}|D_x|^{1/2}u\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \le C\|\varphi\|_{L^2(\mathbf{R}^n)}$$

for $s > 1/2$. (Ben-Artzi and Klainerman [3], *etc).*

By Theorem 4, we have similarly

Corollary 7. *Suppose $\sigma(x,\xi) \sim |\xi|$ and*

$$\sigma(x,\xi) = 0 \text{ on } \Gamma_a.$$

Then the solution to the inhomogeneous equation

$$\begin{cases} (i\partial_t + L)\, u(t,x) = f(t,x), \\ \qquad\qquad u(0,x) = 0 \end{cases}$$

has the estimate

$$\left\|\langle x\rangle^{-1/2+l}\sigma(X,D)u\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \le C\left\|\langle x\rangle^{1/2+l}f\right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)}$$

for $0 < l < 1$ if $n \ge 3$ and $0 < l < 1/2$ if $n = 2$.

Combining Corollaries 6 and 7, we have

Corollary 8. *Let $n \ge 3$, $s, \tilde{s} \ge 0$, and $0 < l < 1$. Suppose*

$$\sigma(x,\xi) \sim |\xi| \quad \text{and} \quad \sigma(x,\xi) = 0 \text{ on } \Gamma_a.$$

Then the solution to the equation

$$\begin{cases} (i\partial_t + L)\, u(t,x) = f(t,x) \\ \qquad\qquad u(0,x) = \varphi(x). \end{cases}$$

has the estimate

$$\begin{aligned} &\left\|\langle x\rangle^{-1/2+l}\sigma(X,D_x)u\right\|_{H^s_t(H^{\tilde{s}}_x)} \\ &\qquad \le C\left\|\langle x\rangle^{l}\langle D_x\rangle^{2s+\tilde{s}+1/2}\varphi\right\|_{L^2(\mathbf{R}^n_x)} + C\left\|\langle x\rangle^{1/2+l}f\right\|_{H^s_t(H^{\tilde{s}}_x)}. \end{aligned}$$

5. Nonlinear Schrödinger Equation

What is the condition of the initial data $\varphi(x)$ for the equation

$$\begin{cases} (i\partial_t + \triangle_x)\, u(t,x) = |\nabla u(t,x)|^N \\ \quad u(0,x) = \varphi(x), \quad t \in \mathbf{R},\ x \in \mathbf{R}^n \end{cases}$$

to have time global solution? Here are some answers:

- $N \ge 3$ (Chihara [4]). Smooth, rapidly decay, and sufficiently small.
- $N \ge 2$ (Hayashi–Miao–Naumkin [5]).$\varphi \in H^{[n/2]+5}$, rapidly decay, and sufficiently small.

Can we weaken the smoothness assumption for $\varphi(x)$? The next results says that the answer is "Yes" if the non-linear term has a *structure* replacing $|\nabla u|^N$ by

$$\left|\left(\frac{x}{\langle x\rangle}\wedge\nabla\right)u\right|^N.$$

Theorem 9. *Suppose $n \geq 3$, $s > (n+3)/2$, and $N \geq 4$. Assume that $\langle x\rangle\langle D_x\rangle^s\varphi \in L^2$ and its L^2-norm is sufficiently small. Then the equation*

$$\begin{cases} (i\partial_t + L)\, u(t,x) = |\sigma(X,D_x)u|^N \\ u(0,x) = \varphi(x), \quad t \in \mathbf{R},\ x \in \mathbf{R}^n, \end{cases}$$

where

$$\begin{cases} \sigma(x,\xi) = 0 \quad on \quad \Gamma_a, \\ \sigma(x,\xi) \sim |\xi|. \end{cases}$$

has a time global solution $u \in C^0(\mathbf{R}_t \times \mathbf{R}^n_x)$.

To prove Theorem 9, we just use Corollary 8 with $f = |\sigma(X,D)u|^N$, but we need to show the estimate

$$\left\| \langle x\rangle^{1/2+l} |\sigma(X,D)u|^N \right\|_{H^s_t(H^{\tilde{s}}_x)} \leq \left\| \langle x\rangle^{-1/2+l} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)}$$

to have an appropriate a priori estimate. The key fact is that the space $H^s_t(H^{\tilde{s}}_x)$ is an algebra if $s > 1/2$ and $\tilde{s} > n/2$. Then we have

$$\begin{aligned} \left\| \langle x\rangle^{1/2+l} |\sigma(X,D_x)u|^N \right\|_{H^s_t(H^{\tilde{s}}_x)} &\leq \left\| \langle x\rangle^{(1/2+l)/N} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)} \\ &\leq \left\| \langle x\rangle^{-1/2+l} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)}. \end{aligned}$$

if $(1/2+l)/N \leq -1/2+l$ or equivalently

$$\frac{N+1}{2(N-1)} \leq l(<1).$$

Thus we have a priori estimate

$$\begin{aligned} &\left\| \langle x\rangle^{-1/2+l} \sigma(X,D_x)u \right\|_{H^s_t(H^{\tilde{s}}_x)} \\ &\qquad \leq C\left\| \langle x\rangle^{l} \langle D_x\rangle^{2s+\tilde{s}+1/2} \varphi \right\|_{L^2(\mathbf{R}^n_x)} + C\left\| \langle x\rangle^{-1/2+l} \sigma(X,D_x)u \right\|^N_{H^s_t(H^{\tilde{s}}_x)}. \end{aligned}$$

6. General idea of the proof of main theorems

The proof of Theorems 3 and 4 consists of the following two procedure:

- First reduce the problem to the case of Laplacian:

$$a(D_x) \quad \Longrightarrow \quad -\Delta,$$

- Then replace $\sigma(X,D)$ by a special operator which commutes with $R(\lambda \pm i0)$:

$$\sigma(X,D) \quad \Longrightarrow \quad \frac{x}{\langle x\rangle} \wedge D.$$

These idea can be realized due to a recent progress on the global L^2-boundedness and the calculus of a class of Fourier integral operators, which was made by Ruzhansky and the author ([9], [10]).

Fourier integral operators

Let

$$Tu(x) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i\phi(x,y,\xi)} p(x,y,\xi) u(y)\, dy d\xi \quad (x \in \mathbf{R}^n),$$

then we have the following Egorov's theory:

- If $\phi(x,y,\xi)$ satisfies (locally) the "graph condition"

  $$\begin{aligned} \Lambda &= \{(x, \phi_x, y, -\phi_y) : \phi_\xi = 0\} \\ &= \{(x,\xi), \chi(x,\xi)\} \subset T^*\mathbf{R}^n \times T^*\mathbf{R}^n \end{aligned}$$

  with some $\chi(x,\xi)$ and $p(x,y,\xi) = 1$, we have (microlocally) the relation

  $$\begin{aligned} &T \cdot A(X,D) = B(X,D) \cdot T + (lower), \\ &B(x,\xi) = (A \circ \chi)(x,\xi). \end{aligned}$$

Hence, by taking a phase function appropriately, properties of the operator $B(X,D)$ can be extracted from those of the well known operator $A(X,D)$. As a special case, if we take

$$\phi(x,y,\xi) = x \cdot \xi - y \cdot \psi(\xi),$$

where

$$\psi(\xi) = \sqrt{a(\xi)} \frac{\nabla a(\xi)}{|\nabla a(\xi)|},$$

we have the exact relation

$$T \cdot (-\triangle) = a(D) \cdot T.$$

If we use this idea to prove main theorems, we need to answer the question when T is globally L^2-bounded.

Here is an previous answer which was used to construct the fundamental solution of Schrödinger equation by "Feynman's path integral".

Theorem 10 (Asada-Fujiwara [2]). *Assume that all the derivatives of $p(x,y,\xi)$ and all the derivatives of each entry of the matrix*

$$D(\phi) = \begin{pmatrix} \partial_x \partial_y \phi & \partial_x \partial_\xi \phi \\ \partial_\xi \partial_y \phi & \partial_\xi \partial_\xi \phi \end{pmatrix}$$

are bounded. Also assume that

$$|\det D(\phi)| \geq C > 0.$$

Then T is $L^2(\mathbf{R}^n)$-bounded.

But unfortunately, our important case

$$\phi(x,y,\xi) = x \cdot \xi - y \cdot \psi(\xi)$$

fails to satisfy the boundedness of $\partial_\xi \partial_\xi \phi$. However if we assume extra decaying properties for phase and amplitude, we can show the (weighted) L^2-boundedness and various calculus:

A class of phase and amplitude functions

We consider the Fourier integral operators of the form

$$T_p u(x) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(x\cdot\xi+\varphi(y,\xi))} p(x,y,\xi)u(y)dyd\xi.$$

with the phase function

$$\phi(x,y,\xi) = x\cdot\xi + \varphi(y,\xi).$$

We assume

$$|\det \partial_y\partial_\xi\varphi(y,\xi)| \geq C > 0$$
$$\left|\partial_y^\alpha\partial_\xi^\beta\varphi(y,\xi)\right| \leq C_{\alpha\beta}\langle y\rangle^{1-|\alpha|}\langle\xi\rangle^{1-|\beta|} \quad (|\beta|\neq 0),$$

but we do not assume the boundedness of

$$\partial_\xi\partial_\xi\phi(x,y,\xi) = \partial_\xi\partial_\xi\varphi(y,\xi).$$

For the amplitude function, we introduce the following classes:

Definition 1. *suppose $m, m', k \in \mathbf{R}$. Amplitude $p(x,y,\xi)$ is of the class $\mathcal{A}_k^{m,m'}$, $\mathcal{R}_k^{m,m'}$ respectively if*

$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma p(x,y,\xi)| \leq C_{\alpha\beta\gamma}\langle x\rangle^{m-|\alpha|}\langle y\rangle^{m'-|\beta|}\langle\xi\rangle^{k-|\gamma|},$$
$$|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma p(x,y,\xi)| \leq C_{\alpha\beta\gamma}\langle x\rangle^{m}\langle y\rangle^{m'-|\beta|}\langle\xi\rangle^{k}.$$

We set

$$\mathcal{A}_k^m = \cup_{m'\in\mathbf{R}}\mathcal{A}_k^{m-m',m'}, \ \mathcal{R}_k^m = \cup_{m'\in\mathbf{R}}\mathcal{R}_k^{m-m',m'}.$$

Boundedness and Calculus

Theorem 11. *Suppose m, $\mu \in \mathbf{R}$. Let $p(x,y,\xi) \in \mathcal{R}_0^m$. Then we have the boundedness*

$$T_p : L^2_{m+\mu}(\mathbf{R}^n) \to L^2_\mu(\mathbf{R}^n).$$

Theorem 12. *Suppose $m, k \in \mathbf{R}$. Let $p(x,y,\xi) \in \mathcal{A}_k^m$. Then we have the decomposition*

$$T_p = T_{p_0} + T_r\,;$$
$$p_0(y,\xi) = p(-\partial_\xi\varphi(y,\xi),y,\xi) \in \mathcal{A}_k^m, \quad r(x,y,\xi) \in \mathcal{R}_{k-1}^{m-1}.$$

Canonical transformation

For the C^∞-diffeomorphism $\psi : \mathbf{R}^n \setminus 0 \to \mathbf{R}^n \setminus 0$ such that $\psi(\lambda\xi) = \lambda\psi(\xi)$ $(\lambda > 0$, $\xi \neq 0)$, we set

$$\begin{aligned} Iu(x) &= F^{-1}\left[Fu(\psi(\xi))\right](x) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(x\cdot\xi - y\cdot\psi(\xi))}u(y)dyd\xi \end{aligned}$$

(neglecting a cut-off function of the origin).

Corollary 13. *Suppose $m \in \mathbf{R}$. Let $a(x,\xi) \in \mathcal{A}_1^m$. We set*

$$\tilde{a}(x,\xi) = a\big(x\psi'(\psi^{-1}(\xi)), \psi^{-1}(\xi)\big).$$

Then we have $\tilde{a}(x,\xi) \in \mathcal{A}_1^m$ and

$$a(X,D)\cdot I = I\cdot\tilde{a}(X,D) + R$$

where

$$R\,:\, L^2_{m-1+\mu}(\mathbf{R}^n) \to L^2_\mu(\mathbf{R}^n)$$

for all $\mu \in \mathbf{R}$.

Change of variables

For the C^∞-diffeomorphism $\kappa : \mathbf{R}^n \setminus 0 \to \mathbf{R}^n \setminus 0$ such that $\kappa(\lambda\xi) = \lambda\kappa(\xi)$ $(\lambda > 0$, $\xi \neq 0)$, we set

$$\begin{aligned} Ju(x) &= (u\circ\kappa)(x) \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(x\cdot\xi - y\cdot\xi)}u(\kappa(y))dyd\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} e^{i(x\cdot\xi - \kappa^{-1}(y)\cdot\xi)}\cdot\left|\det\left(\kappa^{-1}\right)'(y)\right|u(y)dyd\xi \end{aligned}$$

(neglecting a cut-off function of the origin).

Corollary 14. *Suppose $m \in \mathbf{R}$. Let $a(x,\xi) \in \mathcal{A}_1^m$. We set*

$$\tilde{a}(x,\xi) = a\big(\kappa(x), \xi\kappa'(x)^{-1}\big).$$

Then we have $\tilde{a}(x,\xi) \in \mathcal{A}_1^m$ and

$$J\cdot a(X,D) = \tilde{a}(X,D)\cdot J + R,$$

where

$$R\,:\, L^2_{m-1+\mu}(\mathbf{R}^n) \to L^2_\mu(\mathbf{R}^n)$$

for all $\mu \in \mathbf{R}$.

Outline of the proof of Theorems 3 & 4

By using the boundedness and calculus of Fourier integral operators of our class (Corollary 13), we can replace

$$\sigma(X, D) \quad \Longrightarrow \quad \frac{x}{\langle x\rangle} \wedge D$$

in the following sense:

$$\|\sigma(X, D)u\|_{L^2} \leq C\Big(\|(x \wedge D)u\|_{L^2_{m-1}} + \|u\|_{L^2_{m-1}}\Big)$$

where $\sigma(x, \xi) \in \mathcal{A}_1^m$ vanishes on $\Gamma_a = \{(x, \xi) \,:\, x \wedge \xi = 0\}$.

By the change of variables

$$\kappa(x) = \Big(x', \sqrt{x_n^2 - |x'|^2}\Big) \quad x' = (x_1 \ldots, x_{n-1}),$$

for $x = (x_1, x_2 \ldots, x_n)$, it can be reduced to show

$$\|b(X, D)u\|_{L^2} \leq C\Big(\|\Theta(X, D)u\|_{L^2_{m-1}} + \|u\|_{L^2_{m-1}}\Big)$$

by Corollary 14, where $b(x, \xi) = \sigma(\kappa(x), \xi\kappa'(x)^{-1}) \in \mathcal{A}_1^m$ and $\Theta(x, \xi) = \kappa(x) \wedge \xi\kappa'(x)^{-1}$. Note that κ maps the hyperplain $x_n = 1$ to the sphere $|x| = 1$. Here we have

$$\begin{aligned} \Theta_{ij}(x, \xi) &= x_i\xi_j - x_j\xi_i \qquad (i < j < n), \\ \Theta_{in}(x, \xi) &= -\sqrt{x_n^2 - |x'|^2}\xi_i \qquad (i < n) \end{aligned}$$

and, all we have to show is just the estimate

$$\|b(X, D)u\|_{L^2(\mathbf{R}^n)} \leq C\sum_{i=1}^{n-1} \|D_i u\|_{L^2_m(\mathbf{R}^n)}.$$

Since $b(x, \xi) = 0$ on $\{(x, \xi) \,:\, \xi' = 0\}$, we have by Taylor's theorem

$$\begin{aligned} b(x, \xi', \xi_n) &= b(x, 0, \xi_n) + \sum_{i=1}^{n-1} r_i(x, \xi)\xi_i \\ &= \sum_{i=1}^{n-1} r_i(x, \xi)\xi_i, \end{aligned}$$

where

$$r_i(x, \xi) = \int_0^1 (\partial_{\xi_i} b)(x, \theta\xi', \xi_n) \in \mathcal{A}_0^m.$$

Using Theorem 11 to assure the boundedness of $r_i(X, D)$, we have the desired estimate.

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